A Symmetric LPM Model for Mean-Semivariance Optimization

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Abstract

While the semivariance has been variously described as being more in line with investors’ attitude towards risk, implementation has been hampered by computational problems. The original formulation by Markowitz (1959) and Hogan and Warren (1972) requires a laborious iterative process because the cosemivariance matrix is endogenous and a closed form solution does not exist. An exogenous asymmetric cosemivariance matrix computed as in Francis and Archer (1979) and Harlow and Rao (1989) does not always provide a positive semi-definite matrix for which a closed form solution exists. We provide a proof that converts the exogenous asymmetric matrix to a symmetric matrix which allows the mean-semivariance formulation to be solved using Markowitz’s (1959) critical line algorithm. Empirical results demonstrate that the algorithm is robust to a 45 security universe and is still effective at increasing portfolio skewness at a 150 security universe.
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I. Introduction

The semivariance risk measure (LPM degree 2) has been described as being more in line with investors’ attitude towards risk. Markowitz, in particular, has consistently described the semivariance as a more appropriate measure of risk than the variance.\(^1\) Unfortunately, the use of semivariance has been hampered by the complexity of mean-semivariance algorithms. The original formulation by Markowitz (1959) and Hogan and Warren (1972) requires a laborious iterative process because the cosemivariance matrix is endogenous and a closed form solution does not exist. An exogenous asymmetric cosemivariance matrix computed according to Francis and Archer (1979) and Harlow and Rao (1989) does not always provide a positive semi-definite matrix for which a closed form solution exists.

While Jin, Markowitz, and Zhou (2006) recently demonstrate that a closed form solution does exist, many attempts at a closed form solution have been made over the years since Markowitz (1959) originally suggested the measure. Early attempts by Markowitz (1959) and Hogan and Warren (1972) result in an endogenous cosemivariance matrix which can only be solved by computationally intensive algorithms.

Research for a closed form solution has centered on heuristic algorithms that (1) ignore the intercorrelations between securities (Ang, 1975; Nawrocki, 1983; and Markowitz, Todd, Xu and Yumane, 1993) or (2) convert the asymmetric cosemivariance

\(^1\) Markowitz (1959) and Markowitz, Todd, Xu, and Yumane (1993).
matrix to a symmetric cosemivariance matrix that is positive semi-definite (Nawrocki, 1991; Huang, Srivastava, and Raatz, 2001; Estrada, 2007).²

The research reported here takes the latter approach by providing a proof that the asymmetric cosemivariance matrix may be converted to a symmetric cosemivariance matrix without resorting to a heuristic approach. The resulting formulation is an exogenous cosemivariance matrix that is positive semi-definite and for which a closed form solution exists. This allows the application of Markowitz’s critical line algorithm (CLA) to the problem without resorting to the artificial variables that appear in the Markowitz et al. (1993) CLA solution.

The next section presents the issues involved in solving a mean-semivariance optimization problem using the cosemivariance matrix. Then the data and methodology is discussed followed by the empirical results. Finally, conclusions are offered.

II. Mean-Semivariance Optimization

Markowitz (1959) was first to suggest an approach to estimate the semivariance of the portfolio:

\[
S_p = \frac{1}{T} \sum_{k=1}^{K} \left( \sum_{i} X_i R_{ik} \right)^2 = \frac{1}{T} \sum_{k=1}^{K} R_{pk}^2
\]

(1)

where T is the number of observations, K is the number of periods where the portfolio return is below a target return, \(X_i\) is the allocation in security \(i\), \(R_i\) is the return for

² Two other studies by Harlow (1991) and Grootveld and Hallerbach (1999) do not document the formulation of their algorithms.
security $i$, $R_p$ is the return for the portfolio, $R_{ik}$ and $R_{pk}$ are the returns when the portfolio return is below a target return.

The portfolio semivariance is also equal to:

$$S_p = \sum_{i=1}^{n} \sum_{j=1}^{n} X_i X_j CS_{ij}$$

where

$$CS_{ij} = \frac{1}{T} \sum_{k=1}^{K} (R_{ik} - h)(R_{jk} - h)$$

where $CS_{ij}$ is the cosemivariance between securities $i$ and $j$, $h$ is a benchmark return and $K$ is the number of periods where the portfolio underperforms the benchmark. The problem with equations (1)(2)(3) is that the resulting cosemivariance matrix is endogenous – the weights affect the periods in which the portfolio underperforms the benchmark, which in turns affects the computation of the cosemivariance matrix.\(^3\) In other words, the ex-post portfolio semivariance (1) has to be known before we can compute the cosemivariance matrix through (3).

Markowitz (1959) and Hogan and Warren (1972) provide different ways to solve the endogenous cosemivariance matrix problem. Markowitz suggests using Monte Carlo simulation to generate the portfolios. The latter study solves the problem with an iterative Frank-Wolfe algorithm. Both methods require significantly more computer resources than the mean-variance formulation.

To reduce the complexity, heuristics have been offered. Ang (1975) suggests a linear programming approach while Nawrocki (1983) proposes using the semivariance in a further simplification of the Elton, Gruber, and Padberg (1976) constant correlation

\(^3\) Estrada (2007) provides a good explanation of the endogenous cosemivariance problem.
heuristic. Markowitz, Todd, Xu, and Yamane (1993) transform the mean-semivariance problem into a quadratic problem by adding fictitious securities and then solving the problem with the critical line algorithm developed by Markowitz (1959). Recently, de Athayde (2001) proposes a non-parametric approach to calculate the portfolio semivariance. Ballestero (2005) proposes a portfolio semivariance model based on the semivariance below the mean return and then solving it using the Sharpe single index approach.\(^4\) Two other studies by Harlow (1991) and Grootveld and Hallerbach (1999) generate mean-semivariance efficient frontiers but do not provide an explanation as to how they were generated.\(^5\)

Another line of research involves computing a cosemivariance matrix that is exogenous and may be solved by simpler algorithms. Hogan and Warren (1974) and Bawa and Lindenberg (1977) define the cosemivariance between an asset and the market as:

\[
CS_{ij} = \frac{1}{T} \sum_{t=1}^{T} (R_{it} - h)[\text{Min}(R_{mt} - h,0)]
\]

(4)

Francis and Archer (1979) and Harlow and Rao (1989) define the cosemivariance between two securities as:

\[
CS_{ij} = \frac{1}{T} \sum_{t=1}^{T} (R_{it} - h)[\text{Min}(R_{jt} - h,0)]
\]

(5)

The resulting cosemivariance is asymmetric, i.e. \(CS_{ij} \neq CS_{ji}\). This formulation (equations 2 and 5) is not easy to solve as the cosemivariance matrix cannot be guaranteed to be

\(^4\) This formulation is not as useful as Ang and Chua (1979) show that the semivariance below the mean is not consistent with utility theory.

\(^5\) The latter study seems to have used the asymmetric cosemivariance matrix which would have problems in that the matrix will not always be positive semi-definite.
positive semi-definite. Again, heuristic algorithms are used to solve this problem.

Nawrocki (1991) and Huang, Srivastava, and Raatz (2001) suggest a symmetric matrix model based on the correlation coefficient between the two securities $\rho$:

$$CS_{ij} = d_i d_j \rho_{ij}$$

(6)

Where $d_i$ is the semideviation which is the square root of the semivariance $S_i$ for security $i$. The resulting cosemivariance matrix is symmetric and may be solved using simpler algorithms such as the critical line algorithm. Following the same line, Estrada (2007) computes the cosemivariance as follows:

$$CS_{ij} = \frac{1}{T} \sum_{t=1}^{T} [\min(R_{it} - h, 0)][\min(R_{jt} - h, 0)]$$

(7)

Both formulations, (6) and (7) use equation (2) to compute the portfolio semivariance. The latter formulation is a heuristic because both securities $i$ and $j$ have to have returns below the benchmark at the same time. Estrada (2007) demonstrates that this approach provides a very close approximation to the ex post portfolio semivariance (1). The best case would be to use equations (2) and (5) because equation (5) captures all of the interactions between the securities and all of the below target returns.

**A Symmetric Cosemivariance (LPM Degree 2) Model**

The semivariance is a special case of a general family of downside risk measures known as the Lower Partial Moment (LPM). The semivariance is equivalent to the LPM degree 2. Converting the asymmetric LPM matrix to a symmetric LPM matrix provides a

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6 The problem of the asymmetric cosemivariance matrix not being positive semi-definite is related to the number of securities in the matrix (rank of matrix error). Twenty or fewer stocks in the analysis will run into this problem less often than an analysis employing 50 stocks.
7 The LPM risk measures are presented in Bawa (1975), Fishburn (1977), and Harlow and Rao (1989).
positive semi-definite matrix. That the symmetric matrix \( LPM \) leads to the same solution as the asymmetric \( LPM^{(As)} \) is confirmed in the following proof. This proof is valid for a variable number of assets in portfolio as well as for all degrees of the LPM measure.

Proof: Let’s assume some asymmetric matrix \( LPM^{(As)} \), e.g. for two assets:

\[
LPM^{(As)} = \begin{pmatrix} LPM_{11}^{(As)} & LPM_{12}^{(As)} \\ LPM_{21}^{(As)} & LPM_{22}^{(As)} \end{pmatrix}
\]  

(8)

The value of the portfolio Lower partial moment is:

\[
LPM_p^{(As)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} LPM_{11}^{(As)} & LPM_{12}^{(As)} \\ LPM_{21}^{(As)} & LPM_{22}^{(As)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

(9)

\[
= x_1^2 \cdot LPM_{11}^{(As)} + x_2^2 \cdot LPM_{22}^{(As)} + x_1 \cdot x_2 \cdot LPM_{12}^{(As)} + x_1 \cdot x_2 \cdot LPM_{21}^{(As)}
\]

We denote the symmetric matrix \( LPM \) as:

\[
LPM = \begin{pmatrix} LPM_{11} & LPM_{12} \\ LPM_{21} & LPM_{22} \end{pmatrix}
\]

(10)

Then, the value of the portfolio lower partial moment represents:

\[
LPM_p = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} LPM_{11} & LPM_{12} \\ LPM_{21} & LPM_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

(11)

\[
= x_1^2 \cdot LPM_{11} + x_2^2 \cdot LPM_{22} + x_1 \cdot x_2 \cdot LPM_{12} + x_1 \cdot x_2 \cdot LPM_{21}
\]
Let’s define the $LPM$ matrix as:

\[
LPM = \frac{1}{2} \cdot (LPM^{(as)} + LPM^{(as)^T}) = \frac{1}{2} \cdot \left[ \begin{array}{cc}
LP_{11}^{(as)} & LP_{12}^{(as)} \\
LP_{21}^{(as)} & LP_{22}^{(as)}
\end{array} \right] + \left[ \begin{array}{cc}
LP_{11}^{(as)} & LP_{12}^{(as)} \\
LP_{21}^{(as)} & LP_{22}^{(as)}
\end{array} \right]
\]

\[
LPM_{11} = \frac{1}{2} \cdot \left( LPM_{11}^{(as)} + LPM_{11}^{(as)} \right)
\]

\[
LPM_{12} = \frac{1}{2} \cdot \left( LPM_{12}^{(as)} + LPM_{21}^{(as)} \right)
\]

\[
LPM_{21} = \frac{1}{2} \cdot \left( LPM_{21}^{(as)} + LPM_{12}^{(as)} \right)
\]

\[
LPM_{22} = \frac{1}{2} \cdot \left( LPM_{22}^{(as)} + LPM_{22}^{(as)} \right),
\]

(12)

and substitute it into the $LPM_{p}$ (11):

\[
LPM_{p} = \frac{1}{2} \cdot \left[ x_1^2 \cdot (LPM_{11}^{(as)} + LPM_{11}^{(as)}) + x_2^2 \cdot (LPM_{22}^{(as)} + LPM_{22}^{(as)}) \right]
\]

\[
+ x_1 \cdot x_2 \cdot (LPM_{12}^{(as)} + LPM_{21}^{(as)}) + x_1 \cdot x_2 \cdot (LPM_{21}^{(as)} + LPM_{12}^{(as)})
\]

\[
= x_1^2 \cdot LPM_{11}^{(as)} + x_2^2 \cdot LPM_{22}^{(as)} + x_1 \cdot x_2 \cdot LPM_{12}^{(as)} + x_1 \cdot x_2 \cdot LPM_{21}^{(as)}
\]

(13)

Thus, we obtain the same equation as the asymmetric matrix $LPM^{(as)}$ which concludes the proof.

This equivalence is important because a closed form solution is now available for equations (2) and (5) using the critical line algorithm (CLA). Given equation (2) and

\[
E_p = \sum X_i E_i
\]

(14)

the quadratic programming problem may be stated as:

\[
\text{Min } \alpha = S_p - \lambda E_p
\]

s.t.

\[
\sum X_i = 1 \quad X_i \geq 0
\]

(15)

Where $\lambda$ is $dS/dE$, the slope of the efficient frontier at a specific portfolio.
There are other problems with the semivariance equation (2). The first problem is that Grootveld and Hallerbach (1979) note that negative semivariance values may be obtained from the portfolio semivariance equation (2) and the resulting optimization algorithm. As will be noted later in the paper, negative semivariance values are more likely the result of the optimization algorithm attempting to deal with a large number of securities.

Secondly, even though we have the portfolio semivariance equation (2) and Markowitz (1959) states it is equivalent to ex post portfolio semivariance equation (1), the equivalence depends on the weights of the securities in the portfolio. Since the endogenous cosemivariance matrix is not known until the weights are known and the portfolio returns have been determined, equation (1) is not useful for computing efficient frontiers except through computationally intensive iterative techniques. This is the same problem that prevents the computation of stochastic dominance efficient frontiers. By contrast, equations (2) and (5) may be computed ex ante and a closed form solution is available using the asymmetric to symmetric matrix conversion provided above. While Estrada (2007) finds a close correspondence between mean-semivariance portfolios calculated using equations (2) and (7) and the ex-post portfolio semivariance computed using equation (1), we do not. This result is not surprising as it should be clear that the portfolio semivariance computed using (1) is going to have fewer below-target observations than the (2)(5) formulation. In addition, Estrada’s (2)(7) formulation will have fewer below-target observations than the (2)(5) formulation. That is why Estrada found a close correspondence between (1) and (2), i.e. they both use a smaller number of below-target observations. Therefore, no equivalence between equations (1) and (2) should be expected as (5) and (7) are different specifications of the portfolio
semivariance. Our ES algorithm (2)(5) along with Estrada’s ES algorithm (2)(7) have to be considered an optimal formulation given the specification of the portfolio semivariance.\(^8\) However, even if they are a portfolio heuristic, it is not a bad thing – an ex ante closed form solution always exists and is computationally easy. In addition, because of estimation error in the inputs, an optimal portfolio algorithm (mean-variance or mean-semivariance) does not exist. This is because the techniques used to reduce estimation error, resampling and bootstrapping, are always going to result in a heuristic solution.\(^9\)

Moving forward, we may compare the mean-semivariance (ES) efficient frontiers generated using equations (2) and (5) to the mean-variance (EV) efficient frontiers generated by the CLA to see whether the ES CLA algorithm generates an improvement in the ES space over EV CLA generated portfolios.

Because the semivariance implies a skewness preference on the part of investors, we would expect ES portfolios to have higher skewness values than EV portfolios. In addition, Simkowitz and Beedles (1978) find that 92% of the diversifiable skewness in a portfolio is diversified away by as few as five stocks in the portfolio. Therefore, we would expect fewer stocks in an ES portfolio than in an EV portfolio in order for the ES algorithm to generate higher portfolio skewness values than the EV algorithm.

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\(^8\) de Athayde (2001) notes that there are at least three different specifications for the cosemivariance.

\(^9\) Scherer (2002).
III. Methodology

Traditional EV portfolio theory has a number of problems when put into practice. First, the underlying distributions may be non-normal resulting in statistical bias in the inputs. The semivariance helps to reduce this problem because it does not make any distributional assumptions. However, the semivariance is not going to help with the fact that the traditional portfolio optimization algorithm is too powerful for the quality of the inputs and that the algorithm is, in fact, an error maximization process.\(^ {10}\) The major problem is a rank of the matrix error where the number of securities exceeds the number of observations used to estimate the inputs.\(^ {11}\) Markowitz (1987) provides evidence that the CLA is robust relative to the rank of matrix error but there is still a high degree of estimation error due to the fact that a number of small random samples from a stationary distribution will still be all over the map.\(^ {12}\) As a result, Michaud (1989, 1998) developed his “resampled frontier” methodology for mean-variance optimization and Sortino and Forsey (1996) use Efron’s bootstrapping technique to improve the estimate of the semivariance.\(^ {13}\)

For this paper, 150 stocks are randomly selected from the CRSP database with monthly data from January 2001 through December 2006. The order of the stocks was randomly ordered before the optimizations. In order to minimize the effect of estimation error, the CLA and Monte Carlo simulation were used. The simulation uses 300 samples

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\(^ {10}\) Jobson and Korkie (1983), Michaud (1989)

\(^ {11}\) Markowitz (1959) notes that this is analogous to the negative degrees of freedom problem that we would encounter in multiple regression.

\(^ {12}\) Jobson and Korkie (1983)

The results for 300 sequences of 3000 draws indicate that the estimation error for the mean is less than 5% and the estimation error for the standard deviation is less than 1%.

of 3000 random draws from the historic return distribution for each stock. The seed for each sequence of 3000 random draws was also randomly drawn (300 seeds were generated). Both the average of the 300 sequences (Table 1) and the individual 300 sequences (Table 2) are reported in the empirical results. The question to be tested is whether the mean-semivariance algorithm developed earlier generates better mean-semivariance efficient portfolios that the mean-variance algorithm.

IV Empirical Results

Table 1 presents the results of the average mean, variances, covariances, semivariances, and covariances from the 300 sequences. These average values were input into the CLA to generate efficient frontiers for mean-variance (EV) and mean semivariance (ES). The portfolio universe is increased from 25 to 150 securities in order to see when the CLA starts to break down due to the large number of securities. Both algorithms start to generate negative portfolio semivariance values (equation 2) at 50 securities. The ex-post portfolio semivariance value (equation 1) never turns negative. Because negative semivariance portfolios are considered to be a problem with both algorithms, portfolios with negative portfolio semivariances (2) are not used to generate the results in Table 1 or Table 2. The ES portfolios are expected to exhibit increased skewness when compared to the EV portfolios and this is observed from the 25 security to 150 security universes. The average ES portfolio has a higher skewness than the average EV portfolio. Because skewness is diversified away with fewer stocks than
variance, we would expect the average ES portfolio to have fewer stocks than the average EV portfolio. With the exception of the 25 security universe, this is true until the 100 security universe is reached. We would expect the average R/SV ratio for the ES portfolio to have a higher R/SV ratio than the EV portfolio. This is generally true for the R/SV ratio computed from the ex-post portfolio semivariance (1) until the universe is larger than 110 securities. The exceptions occur at 50, 60, 85, and 90 securities. The result using the R/SV ratio computed using (2) is comparable with the exceptions occurring at 65, 70, and 80 securities.
Table 1 – Aggregate Results for E-V and E-LPM Degree 2 Portfolios for 300 Sequences of 3000 Draws Each for Security Universes of 25 to 150 Securities

<table>
<thead>
<tr>
<th>Method</th>
<th>Number Of Securities</th>
<th>Average Skewness</th>
<th>Average Number of Securities</th>
<th>Average R/SV Ratio Eq. (1)</th>
<th>Average R/SV Ratio Eq. (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-LPM</td>
<td>25</td>
<td>-0.4206</td>
<td>7.13</td>
<td>.6229</td>
<td>0.7429</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.6565</td>
<td>6.54</td>
<td>.5789</td>
<td>0.6866</td>
</tr>
<tr>
<td>E-LPM</td>
<td>30</td>
<td>-0.2426</td>
<td>5.10</td>
<td>.5883</td>
<td>0.8044</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.7044</td>
<td>8.61</td>
<td>.5718</td>
<td>0.7009</td>
</tr>
<tr>
<td>E-LPM</td>
<td>35</td>
<td>-0.3399</td>
<td>7.07</td>
<td>.6060</td>
<td>0.8251</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.6810</td>
<td>9.54</td>
<td>.5624</td>
<td>0.6751</td>
</tr>
<tr>
<td>E-LPM</td>
<td>40</td>
<td>0.0874</td>
<td>7.39</td>
<td>.7158</td>
<td>1.2104</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.3549</td>
<td>8.64</td>
<td>.6857</td>
<td>0.9843</td>
</tr>
<tr>
<td>E-LPM</td>
<td>45</td>
<td>0.0997</td>
<td>7.54</td>
<td>.7142</td>
<td>1.2082</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.4591</td>
<td>8.86</td>
<td>.7053</td>
<td>1.0159</td>
</tr>
<tr>
<td>E-LPM</td>
<td>50</td>
<td>0.5426</td>
<td>6.36</td>
<td>.8871</td>
<td>2.4195</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.2914</td>
<td>10.57</td>
<td>.9315</td>
<td>2.2057</td>
</tr>
<tr>
<td>E-LPM</td>
<td>55</td>
<td>0.7725</td>
<td>6.20</td>
<td>.8770</td>
<td>2.0407</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>0.4085</td>
<td>11.28</td>
<td>.8620</td>
<td>2.0325</td>
</tr>
<tr>
<td>E-LPM</td>
<td>60</td>
<td>0.5455</td>
<td>6.83</td>
<td>.8910</td>
<td>2.4203</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.1509</td>
<td>9.22</td>
<td>.9015</td>
<td>2.1167</td>
</tr>
<tr>
<td>E-LPM</td>
<td>65</td>
<td>0.6205</td>
<td>5.70</td>
<td>.8256</td>
<td>1.2058</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>0.1103</td>
<td>9.12</td>
<td>.7725</td>
<td>1.3693</td>
</tr>
<tr>
<td>E-LPM</td>
<td>70</td>
<td>0.6210</td>
<td>5.30</td>
<td>.8305</td>
<td>1.2175</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>0.0929</td>
<td>9.41</td>
<td>.7768</td>
<td>1.3800</td>
</tr>
<tr>
<td>E-LPM</td>
<td>75</td>
<td>0.7680</td>
<td>5.60</td>
<td>.8388</td>
<td>1.9163</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>0.1241</td>
<td>9.19</td>
<td>.8186</td>
<td>1.4847</td>
</tr>
<tr>
<td>E-LPM</td>
<td>80</td>
<td>0.6429</td>
<td>5.33</td>
<td>.8266</td>
<td>1.6412</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>0.0779</td>
<td>9.19</td>
<td>.8227</td>
<td>2.5367</td>
</tr>
<tr>
<td>E-LPM</td>
<td>85</td>
<td>0.5886</td>
<td>5.60</td>
<td>.8794</td>
<td>1.5955</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.0199</td>
<td>7.46</td>
<td>.8905</td>
<td>1.3655</td>
</tr>
<tr>
<td>E-LPM</td>
<td>90</td>
<td>0.6494</td>
<td>5.67</td>
<td>.8326</td>
<td>1.5380</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>0.0105</td>
<td>7.93</td>
<td>.8728</td>
<td>1.3697</td>
</tr>
<tr>
<td>E-LPM</td>
<td>95</td>
<td>0.6085</td>
<td>6.09</td>
<td>.8879</td>
<td>1.6802</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>0.2270</td>
<td>4.43</td>
<td>.8025</td>
<td>1.0203</td>
</tr>
<tr>
<td>E-LPM</td>
<td>100</td>
<td>0.6305</td>
<td>6.17</td>
<td>.8769</td>
<td>1.5751</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>0.2286</td>
<td>4.88</td>
<td>.7943</td>
<td>1.0237</td>
</tr>
<tr>
<td>E-LPM</td>
<td>110</td>
<td>0.4441</td>
<td>6.64</td>
<td>.9131</td>
<td>1.7490</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.0010</td>
<td>5.13</td>
<td>.8927</td>
<td>1.3529</td>
</tr>
<tr>
<td>E-LPM</td>
<td>120</td>
<td>0.4184</td>
<td>6.55</td>
<td>.9339</td>
<td>1.6460</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.0797</td>
<td>5.33</td>
<td>.9508</td>
<td>2.1989</td>
</tr>
<tr>
<td>E-LPM</td>
<td>130</td>
<td>0.4203</td>
<td>6.28</td>
<td>.9287</td>
<td>1.6127</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.1215</td>
<td>6.30</td>
<td>.9832</td>
<td>2.8488</td>
</tr>
<tr>
<td>E-LPM</td>
<td>140</td>
<td>0.2229</td>
<td>6.83</td>
<td>.8930</td>
<td>1.6017</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.0355</td>
<td>5.56</td>
<td>.9038</td>
<td>1.3850</td>
</tr>
<tr>
<td>E-LPM</td>
<td>150</td>
<td>0.2556</td>
<td>6.67</td>
<td>.8404</td>
<td>1.5399</td>
</tr>
<tr>
<td>E-V</td>
<td></td>
<td>-0.0205</td>
<td>5.33</td>
<td>.8583</td>
<td>1.3889</td>
</tr>
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</table>
In Table 2, the average skewness, the average number of securities, the average R/SV ratios for each of the 300 sequences are paired ES to EV and tested to see if the differences (ES-EV) are significantly different from zero using a t-test. The average skewness is greater for the ES portfolios for all security universes with the result being significant from the 35 security to the 150 security universe. The ES portfolios have significantly fewer securities than the EV portfolios from 30 securities to 90 securities with the exception of the 40 and 45 security universes. The R/SV ratio (1) is significantly higher for the ES portfolios for the 25-40 security universes and the R/SV ratio (2) is significantly higher for the 25-45 security universes. The last column in Table 2 reports the minimum portfolio semivariance value. The ES algorithm is more efficient than the EV algorithm to 50 securities. From 55 to 80 securities, it is less efficient and then becomes more efficient from 85 to 110 securities.

These results indicate the ES algorithm is pretty robust for general use with security universes of 40-45 securities. If resampling is used, the ES algorithm may be used for as many as 110 securities.
Table 2 – T-test Results Between the Differences of 300 Sequences for Portfolio Universe Sizes of 25 to 150 Securities

<table>
<thead>
<tr>
<th>Method</th>
<th>Number Of Securities</th>
<th>Difference Skewness</th>
<th>Difference # Securities</th>
<th>Difference R/SVr Ratio (1)</th>
<th>Difference R/SVr Ratio (2)</th>
<th>Minimum SV Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-LPM E-V</td>
<td>25</td>
<td>0.941</td>
<td>0.021</td>
<td>2.630*</td>
<td>1.804*</td>
<td>.000157</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>30</td>
<td>1.593</td>
<td>-3.278*</td>
<td>1.854*</td>
<td>1.670*</td>
<td>.000073</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>35</td>
<td>1.749*</td>
<td>-5.610*</td>
<td>4.208*</td>
<td>4.198*</td>
<td>.000076</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>40</td>
<td>7.334*</td>
<td>-1.029</td>
<td>8.477*</td>
<td>8.837*</td>
<td>.000032</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>45</td>
<td>5.306*</td>
<td>-0.995</td>
<td>0.610</td>
<td>1.968*</td>
<td>.000022</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>50</td>
<td>10.466*</td>
<td>-9.946*</td>
<td>-1.694*</td>
<td>-0.754</td>
<td>.000009</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>55</td>
<td>7.047*</td>
<td>-6.021*</td>
<td>0.523</td>
<td>-0.576</td>
<td>.000026</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>60</td>
<td>10.198*</td>
<td>-4.795*</td>
<td>0.073</td>
<td>-1.173</td>
<td>.000012</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>65</td>
<td>3.566*</td>
<td>-2.278*</td>
<td>1.607</td>
<td>-0.700</td>
<td>.000093</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>70</td>
<td>3.660*</td>
<td>-2.593*</td>
<td>1.769*</td>
<td>-0.888</td>
<td>.000096</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>75</td>
<td>3.578*</td>
<td>-2.956*</td>
<td>0.044</td>
<td>-5.345*</td>
<td>.000014</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>80</td>
<td>4.475*</td>
<td>-1.785*</td>
<td>-0.760</td>
<td>-1.654</td>
<td>.000012</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>85</td>
<td>4.521*</td>
<td>-1.857*</td>
<td>-1.028</td>
<td>-0.291</td>
<td>.000020</td>
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<tr>
<td>E-LPM E-V</td>
<td>90</td>
<td>5.700*</td>
<td>-5.641*</td>
<td>-2.080*</td>
<td>-1.593</td>
<td>.000016</td>
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<tr>
<td>E-LPM E-V</td>
<td>95</td>
<td>4.864*</td>
<td>1.675*</td>
<td>1.362</td>
<td>0.054</td>
<td>.000019</td>
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<tr>
<td>E-LPM E-V</td>
<td>100</td>
<td>3.483*</td>
<td>0.825</td>
<td>1.526</td>
<td>0.908</td>
<td>.000060</td>
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<tr>
<td>E-LPM E-V</td>
<td>110</td>
<td>3.061*</td>
<td>0.425</td>
<td>0.041</td>
<td>0.169</td>
<td>.000018</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>120</td>
<td>3.910*</td>
<td>1.644</td>
<td>1.120</td>
<td>0.377</td>
<td>.000015</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>130</td>
<td>4.213*</td>
<td>0.649</td>
<td>0.132</td>
<td>0.322</td>
<td>.000018</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>140</td>
<td>4.485*</td>
<td>-0.149</td>
<td>-0.896</td>
<td>-0.514</td>
<td>.000052</td>
</tr>
<tr>
<td>E-LPM E-V</td>
<td>150</td>
<td>4.197*</td>
<td>-0.200</td>
<td>-0.911</td>
<td>-0.457</td>
<td>.000059</td>
</tr>
</tbody>
</table>
In Figures 1-4, the efficient frontiers for security universes of 25 to 150 securities are plotted. The ELPMf and EVf frontiers result from the ex-ante estimation of the portfolio semivariance (equation 2). The ELPMr and EVr frontiers represent the ex-post calculation of the portfolio semivariance using equation 1. In Figure 1, the efficient frontiers are computed using universe sizes of 25 and 45 securities. All of the frontiers are concave and are reasonably continuous. The ELPM portfolios are more E-S efficient than the EV portfolios.

It is in Figure 2 with the 50 and 55 security universes that the negative portfolio semivariance is first encountered. The ex-ante frontiers (ELPMf and EVf) are concave and continuous, however, the negative semivariance is evident in the low risk-low return portfolios. The ELPMf frontier is more E-S efficient than the EVf frontier. The ex-post portfolios do not exhibit negative semivariance, however, the ELPMr frontiers are not clearly more E-S efficient than the EVr frontiers. The ex-post frontiers have a kink in them that we would not expect from an efficient frontier. These results are repeated in Figure 3 with the 65 and 85 security universes.

In Figure 4, the ELPM algorithm has essentially broken down by 100 and 150 security universes. The ELPMf frontier is still concave but the ex-post ELPMr frontier becomes inefficient when the negative ex-ante portfolio semivariances appear. It seems clear that the CLA algorithm is starting to break down at 50 securities, however, the ELPM algorithm continues to provide additional skewness to the portfolios when compared to the EV algorithm, even at 150 securities.
Figure 1 – E-LPM and E-V Efficient Frontiers for 25 and 45 Security Universes
Figure 2 – E-LPM and E-V Efficient Frontiers for 50 and 55 Security Universes
Figure 3 – E-LPM and E-V Efficient Frontiers for 65 and 85 Security Universes
Figure 4 – E-LPM and E-V Efficient Frontiers for 100 and 150 Security Universes
V. Summary and Conclusions

A symmetric cosemivariance matrix portfolio selection model that is equivalent to the asymmetric cosemivariance matrix model has been presented. It has the virtue of providing a closed form solution that is easily solved by Markowitz’s critical line algorithm (CLA). While the CLA is an optimization algorithm, portfolio theory limits its use to a heuristic algorithm because of the resampling techniques used to reduce estimation error in the inputs.

In an empirical test that uses 300 sequences of 3000 random numbers to reduce estimation error, the symmetric cosemivariance model exhibits an ability to increase positive skewness in a portfolio when compared to portfolios generated by the traditional mean-variance model. This is the major benefit of the model, i.e. it provides better mean-semivariance portfolios than a mean-variance algorithm. The results are pretty robust until the security universe increases past 50 securities and inconsistencies start to show up. When working with security universes up to 40-45 securities, the mean-semivariance model is superior to the mean-variance model. At 50 securities or larger, both algorithms start to provide negative semivariance values which can be construed as evidence that the algorithms are breaking down due to the large number of securities. However, the negative semivariance portfolios are part of an ex-ante concave mean-semivariance frontier even when a 150 security universe is used.

Another issue is that the portfolio semivariance computed from the cosemivariance matrix rarely agrees with the portfolio semivariance computed from portfolio returns. Since we don’t know the portfolio returns until the portfolio is determined, this ex-post semivariance formulation is very difficult to implement in a portfolio selection algorithm. A closed form
mean-semivariance solution is not available except for various iterative techniques that are essentially trial and error searches. As a result, various heuristic algorithms have been proposed to provide a closed form solution when using the semivariance as a risk measure. A major benefit of the symmetric cosemivariance matrix model when compared to other mean-semivariance heuristics is that it preserves the intercorrelations between securities, thus providing true diversification.

Even though the semivariance values are different, efficient frontiers using the portfolio semivariance computed from the symmetric cosemivariance matrix show a strong correspondence to efficient frontiers using the ex-post portfolio semivariance computed from the portfolio returns until the security universe increases to 100 securities and beyond.
References


